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Note on fluctuating flow near a stagnation point

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The problem of the flow near a stagnation point when the main stream outside the boundary layer fluctuates in magnitude but not in direction about a steady mean is discussed. The velocity distribution is found in the two limiting cases of small and large values of the frequency of the oscillation. The corresponding two approximate solutions give similar results in an overlapping range of frequency.

1. Introduction

The laminar boundary layer in two-dimensional flow, when the velocity of the oncoming flow relative to the body oscillates in magnitude but not in direction, has been investigated by Lighthill (1954). He found that for frequencies greater than a certain value, the oscillations within the boundary layer are close to shear waves unaffected by the mean flow. For frequencies less than this value the oscillations are closely approximated by the sum of the parts corresponding to the instantaneous velocity and the acceleration of the oncoming flow.

Froessling (see Schlichting 1955) has obtained an exact solution for the steady flow of an incompressible viscous liquid near a stagnation point. Ratna & Rajeshwari (1962) have solved the similar problem for a visco-elastic liquid by a Kármán–Pohlhausen method from which the corresponding results for ordinary viscous fluids can be derived. In this paper such a problem is discussed for a viscous liquid when the main stream outside the boundary layer fluctuates in magnitude but not in direction about a steady mean. The method of analysis is the same as that of Lighthill for the problem mentioned above. The velocity components in the directions of r and z (r, θ, z are cylindrical-polar co-ordinates) outside the boundary layer are taken as

$$U = a_0(1+\epsilon e^{i\omega t})r \quad \text{and} \quad W = -2a_0(1+\epsilon e^{i\omega t})z,$$

respectively, where a_0 is a constant, $z = 0$ is a wall, the origin is the stagnation point, and ϵ and ω are the amplitude and the frequency of the oscillation, respectively. Taking the result for the steady part of flow from Ratna & Rajeshwari (1962), the unsteady part of the flow has been calculated for small and for large values of ω. It is shown that the two approximate solutions overlap for

$$\omega = 8{\cdot}0846a_0.$$

2. Equations of motion

The equations of axisymmetric motion for viscous incompressible liquid in cylindrical-polar co-ordinates, when the velocity perpendicular to the meridian plane is zero, are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right], \tag{1}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{2}$$

where all symbols have their usual fluid dynamical significance. The equation of continuity is

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0. \tag{3}$$

Consider a steady stream of viscous liquid impinging on a wall $z = 0$ and flowing away radially in all directions. The stagnation point is at the origin and the flow is in the direction of the negative z-axis. The velocity components in the directions of r and z are respectively $U = ar$ and $W = -2az$ outside the boundary-layer region. This suggests the form of the velocity components within the boundary-layer region as

$$u = r\,d\{f(z)\}/dz, \quad w = -2f(z).$$

The boundary conditions are

$$\begin{aligned} u &= 0,\ w = 0 \quad \text{at} \quad z = 0,\\ u &= U \qquad\quad\ \text{at} \quad z = \delta, \end{aligned} \tag{4}$$

where δ is the boundary-layer thickness.

Now we consider that a oscillates about a steady mean value a_0 and at any time is given by

$$a = a_0(1+\epsilon e^{i\omega t}), \tag{5}$$

where ϵ is small. The velocity components in the directions of r and z outside the boundary-layer region are given by

$$U = a_0(1+\epsilon e^{i\omega t})\,r, \quad W = -2a_0(1+\epsilon e^{i\omega t})\,z, \tag{6}$$

which means that the main stream velocity is always inclined at an angle $\tan^{-1}(-2z/r)$ to the wall and oscillates in magnitude only. We suppose that the velocity components and the pressure within the boundary-layer region are performing small oscillations about a steady mean and write $f(z)$ and $p(r,z)$ in the following form:

$$f(z) = (a_0\nu)^{\frac{1}{2}}\,[\phi_0(\eta)+\epsilon\phi_1(\eta)\,e^{i\omega t}], \tag{7}$$

$$p(r,z) = p_0(r,z)+\epsilon p_1(r,z)\,e^{i\omega t}, \tag{8}$$

where $\eta = z/\delta$. The boundary conditions (4) become

$$\left.\begin{aligned} &\phi_0' = \phi_0 = 0, \quad \phi_1' = \phi_1 = 0 \quad \text{at} \quad \eta = 0,\\ &\phi_0' = D, \quad \phi_1' = D, \quad \phi_0'' = \phi_0''' = \phi_0^{\text{iv}} = 0, \quad \phi_1'' = \phi_1''' = \phi_1^{\text{iv}} = 0 \quad \text{at} \quad \eta = 1, \end{aligned}\right\} \tag{9}$$

where $D = (a_0/\nu)^{\frac{1}{2}}\,\delta$. For steady flow $D = 2{\cdot}5494$ and $\phi_0(\eta)$ is given by (see Ratna & Rajeshwari 1962)

$$\phi_0(\eta) = D(1{\cdot}6562\eta^2 - 1{\cdot}0832\eta^3 - 0{\cdot}0314\eta^4 + 0{\cdot}3501\eta^5 - 0{\cdot}1146\eta^6). \qquad (10)$$

Here a prime denotes differentiation with respect to η. Substituting (7), (8) in (1) and (2) and neglecting the terms containing square and higher orders of ϵ, we get

$$i\Omega D^2\phi_1' + 2D\phi_0'\phi_1' - 2D(\phi_1''\phi_1 + \phi_0\phi_1'') - \phi_1''' = -(D^3/\rho a_0^2 r)\,\partial p_1(r,z)/\partial r, \qquad (11)$$

$$-2i\Omega D^2\phi_1 + 4D(\phi_0'\phi_1 + \phi_0\phi_1') + \phi_1'' = -(D^2/\mu a_0)\,\partial p_1(r,z)/\partial\eta. \qquad (12)$$

The condition of integrability of (11) and (12) is

$$i\Omega D^2\phi_1' + 2D\phi_0'\phi_1' - 2D(\phi_0''\phi_1 + \phi_0\phi_1'') - \phi_1''' = C, \qquad (13)$$

where C is a constant of integration. The boundary conditions at $\eta = 1$ from (9) give $C = 2D^3 + i\Omega D^3$, and equation (13) becomes

$$i\Omega D^2(\phi_1' - D) + 2D(\phi_0'\phi_1' - D^2) - 2D(\phi_0''\phi_1 + \phi_0\phi_1'') - \phi_1''' = 0. \qquad (14)$$

3. Low frequency

The solution of equation (14) in the limiting case $\omega \to 0$ is the quasi-steady solution; let it be denoted by $\phi_2(\eta)$. Following Lighthill (1954) it can be written as

$$\phi_2(\eta) = \tfrac{1}{2}[\phi_0(\eta) + \eta\phi_0']. \qquad (15)$$

Substituting the expression for ϕ_0 from (10), we get

$$\phi_2(\eta) = D(2{\cdot}4843\eta^2 - 2{\cdot}1664\eta^3 - 0{\cdot}0785\eta^4 + 1{\cdot}0503\eta^5 - 0{\cdot}4001\eta^6). \qquad (16)$$

For general values of ω we write

$$\phi_1(\eta) = \phi_2(\eta) + i\phi_3(\eta). \qquad (17)$$

Substituting (17) in (14) and using the fact that $\phi_2(\eta)$ is a solution for $\omega = 0$, we get the equation for $\phi_3(\eta)$ as

$$D^2(\omega/a_0)(\phi_2' - D) + 2D\phi_0'\phi_3' - 2D(\phi_0''\phi_3 + \phi_0\phi_3'') - \phi_3''' = 0. \qquad (18)$$

Boundary conditions on $\phi_3(\eta)$ are

$$\left.\begin{array}{llll} \phi_3 = 0, & \phi_3' = 0, & \phi_3''' = -\Omega D^3 & \text{at} \quad \eta = 0, \\ \phi_3' = 0, & \phi_3'' = 0, & \phi_3''' = 0, \quad \phi_3^{\text{iv}} = 0 & \text{at} \quad \eta = 1, \end{array}\right\} \qquad (19)$$

where $\Omega = \omega/a_0$. Representing $\phi_3(\eta)$ by a seventh-degree polynomial in η, it can be written as

$$\phi_3(\eta) = \tfrac{1}{30}\Omega D^3\eta^2(1-\eta)^5 - b[14\eta^2(1-\eta)^5 + 7\eta(1-\eta)^6 + (1-\eta)^7 - 1], \qquad (20)$$

which satisfies (19). Integrating (18) from $\eta = 0$ to $\eta = 1$ and applying (19), we get

$$\Omega D^2\phi_2(1) + 6D\int_0^1 \phi_1'(\eta)\,\phi_3'(\eta)\,d\eta - 2D^2\phi_3(1) + \phi_3''(0) - \Omega D^3 = 0. \qquad (21)$$

Substituting the expressions for $\phi_0(\eta)$ from (10), $\phi_2(\eta)$ from (16) and $\phi_3(\eta)$ from (20) in (21), we get $b = 0{\cdot}033008$. Hence the expression for $\phi_3(\eta)$ can be written as

$$\phi_3(\eta) = \Omega(0{\cdot}7834\eta^2 - 2{\cdot}6617\eta^3 + 4{\cdot}3684\eta^4 - 3{\cdot}6752\eta^5 + 1{\cdot}6066\eta^6 - 0{\cdot}2883\eta^7). \quad (22)$$

This solution is taken only as far as the first power in Ω and further powers can be calculated if required.

4. High frequency

For high frequency, i.e. when Ω is greater than some as yet undetermined value, the above treatment will not give a correct picture. For this case equation (14) is approximated by retaining terms involving Ω and the derivative of highest order. Then equation (14) reduces to

$$\phi_1''' = i\Omega D^2(\phi_1' - D). \quad (23)$$

The boundary conditions on $\phi_1(\eta)$ are

$$\left.\begin{array}{lll} \phi_1 = 0, \quad \phi_1' = 0 & \text{at} & \eta = 0, \\ \phi_1' \to D & \text{as} & \eta \to \infty. \end{array}\right\} \quad (24)$$

The solution of (23) satisfying (24) is

$$\phi_1(\eta) = D\eta - \Omega^{-\frac{1}{2}}[(1-i)/\sqrt{2} - \exp\{-(1+i)\,D\eta(\tfrac{1}{2}\Omega)^{\frac{1}{2}} - \tfrac{1}{4}i\pi\}], \quad (25)$$

which shows that the oscillations of the velocity components within the boundary layer are to a close approximation 'shear waves' unaffected by the mean flow.

5. Discussion

The expression for the skin friction at the wall is given by $\mu(\partial u/\partial z)\, z = 0$. In the case of high frequency it is

$$\mu(a_0\nu)^{\frac{1}{2}}\,(r/D^2)\,[3{\cdot}3124D + \epsilon D^2(\tfrac{1}{2}\Omega)^{\frac{1}{2}}\,(1+i)\,e^{i\omega t}]. \quad (26)$$

The amplitude of its fluctuation increases with Ω and its phase is ahead of the fluctuation of the main stream by 45°. In the case of low frequency the skin friction at the wall is

$$\mu(a_0\nu)^{\frac{1}{2}}\,(r/D^2)\,[3{\cdot}3124D + \epsilon(4{\cdot}9686D + i1{\cdot}5668\Omega)\,e^{i\omega t}]. \quad (27)$$

It has a phase lead of $\tan^{-1}(\Omega/8{\cdot}0846)$ over the oscillation of the main stream. This phase lead increases with Ω and becomes 45° for $\Omega = 8{\cdot}0846$ which is its phase lead in the case of high frequency. For this value of Ω the amplitude of its oscillation in both the cases are approximately the same. Hence this is the value of Ω at which transition from one type of flow to the other occurs. Denoting the real and imaginary parts of u_1 by u_r and u_i respectively, the graphs of u_r/a_0r and u_i/a_0r have been plotted against η for $\Omega = 8{\cdot}0846$ in figure 1. The continuous lines indicate the values taken from the high-frequency solution and the broken lines indicate that from the low-frequency solution. Similar graphs for the real and imaginary parts of w_1 have been drawn in figure 2. Both the graphs reveal that the unsteady part of the radial as well as the axial component of the velocity

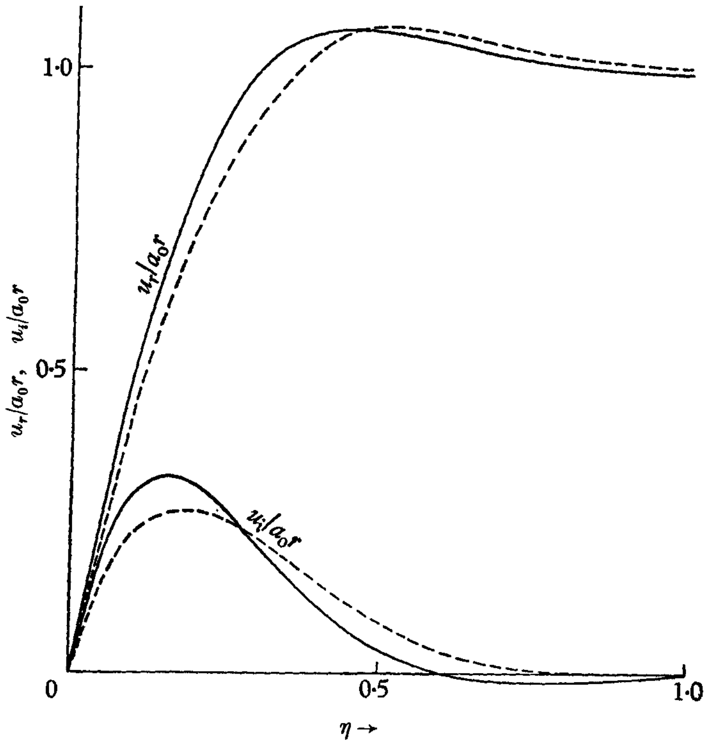

FIGURE 1. Approximations to u_r/a_0r and u_i/a_0r as functions of η: —, high frequency; ---, low frequency.

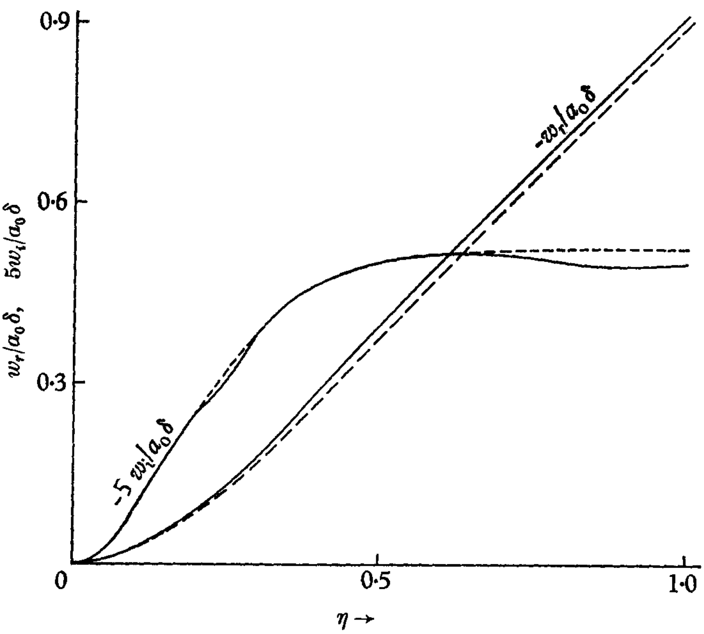

FIGURE 2. Approximations to $w_r/a_0\delta$ and $w_i/a_0\delta$ as functions of η: —, high frequency; ---, low frequency.

within the boundary-layer region from the two approximate solutions almost overlap at this frequency of oscillation of the main flow.

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REFERENCES

LIGHTHILL, M. J. 1954 *Proc. Roy. Soc.* A, **224**, 1.

RATNA, S. L. & RAJESHWARI, G. K. 1962 *Z. Angew. Math. Phys.* **13**, 43.

SCHLICHTING, H. 1955 *Boundary-Layer Theory*, p. 73. London: Pergamon Press.